The QCD Phase Diagram from Lattice Simulations

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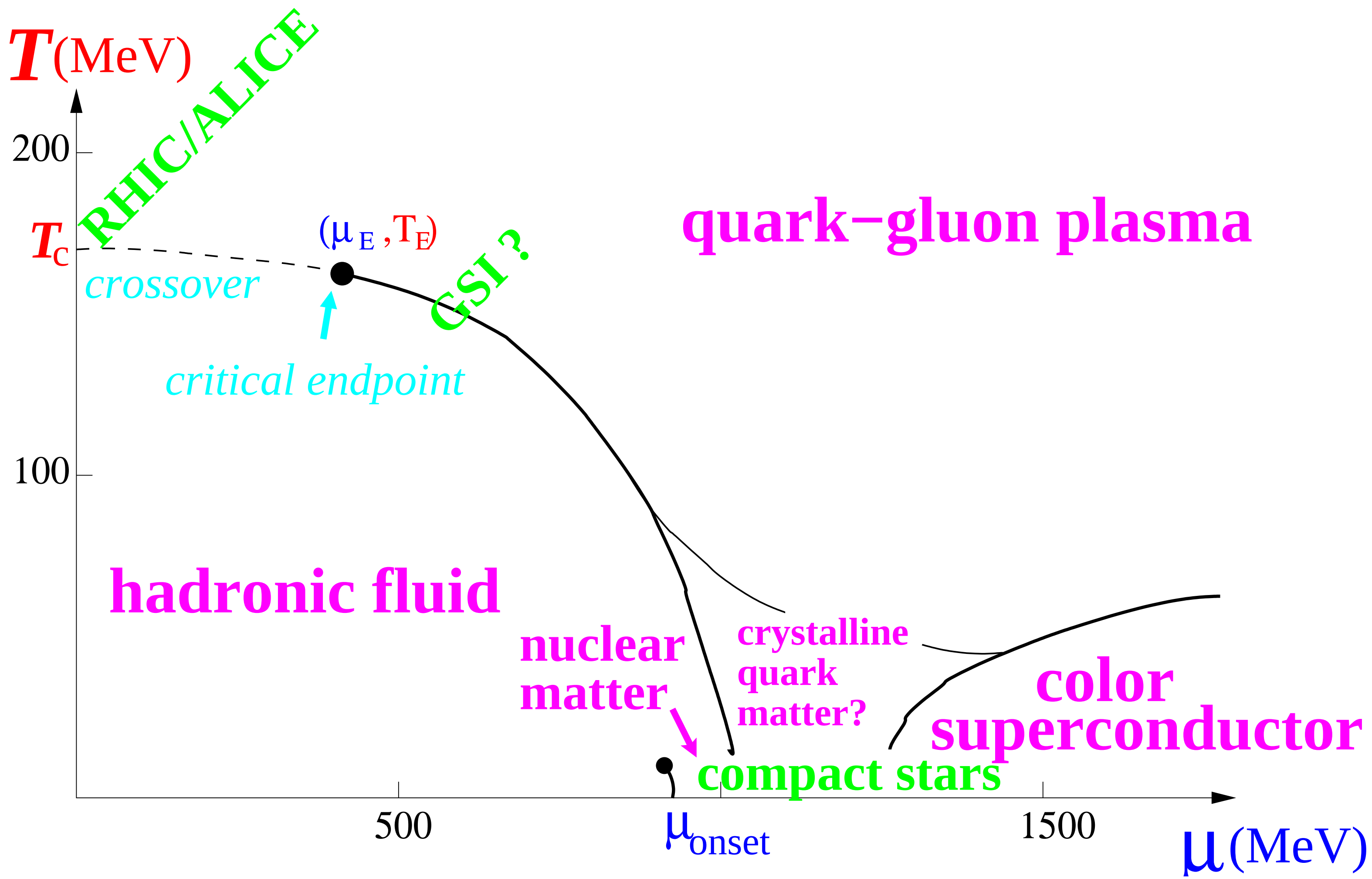
- Difficulties at $\mu \neq 0$
- Progress at small μ/T
- Taylor Expansion of the Free Energy
- Color Superconductivity
- Superfluidity in the NJL model

Daresbury 3^{rd} March

The QCD Phase Diagram



Bluffer's Guide to Lattice QCD

Feynman Path Integral for QCD

$$\langle\mathcal{O}(\psi,\bar{\psi},A_\mu)\rangle = \frac{1}{Z}\int D\psi D\bar{\psi} DA_\mu \mathcal{O} e^{\frac{i}{\hbar}\int_x(\bar{\psi}M[A_\mu]\psi+\frac{1}{4}F_{\mu\nu}F^{\mu\nu})}$$

with $Z \equiv \langle 1\rangle$

Two technical tricks:

- Analytically continue from Minkowski to Euclidean space $t \mapsto ix_4$ *FPI has better convergence properties*
- Discretise $F_{\mu\nu}$ and M on a 4d spacetime lattice *FPI becomes an ordinary multi-dimensional integral*

$\langle\mathcal{O}\rangle$ can now be estimated numerically using Monte Carlo importance sampling, in effect "simulating" quantum fluctuations of the the $\psi,\bar{\psi}$ and A_μ fields

States can be analysed by choosing $\mathcal{O}$ with appropriate quantum numbers and then measuring the energy via decay in Euclidean time:

$$\langle \mathcal{O}(0)\mathcal{O}^\dagger(x_4)\rangle \propto e^{-Ex_4}$$

Thermal effects modelled by restricted the time extent of the Euclidean universe to $0 < x_4 < \beta \Rightarrow$

Z includes all excitations with Boltzmann weight $e^{-\beta E}$, ie.

$$\langle \mathcal{O}\rangle = \frac{1}{Z}\sum_i \mathcal{O}_i e^{-\frac{E_i}{kT}}$$

with temperature $T = \beta^{-1}$

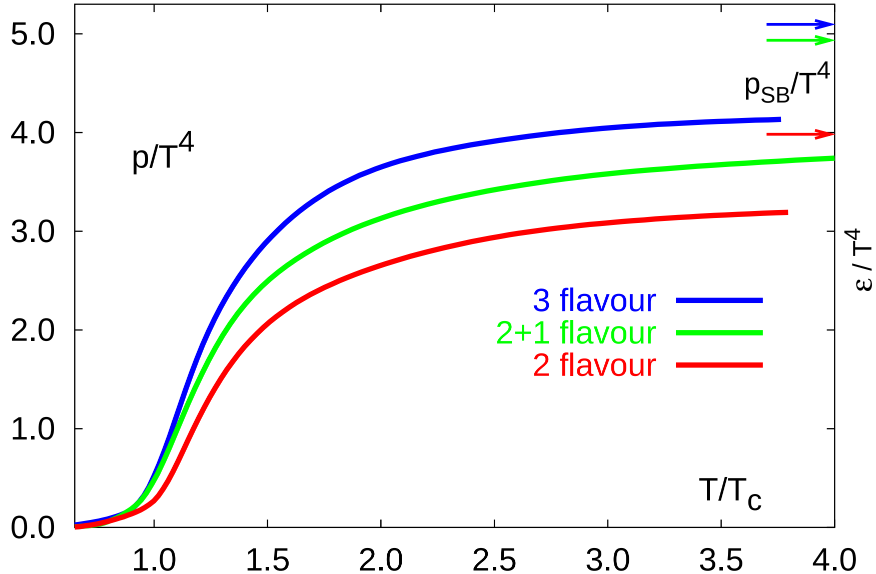
Lattice currently the most systematic way of studying QCD with $T > 0$

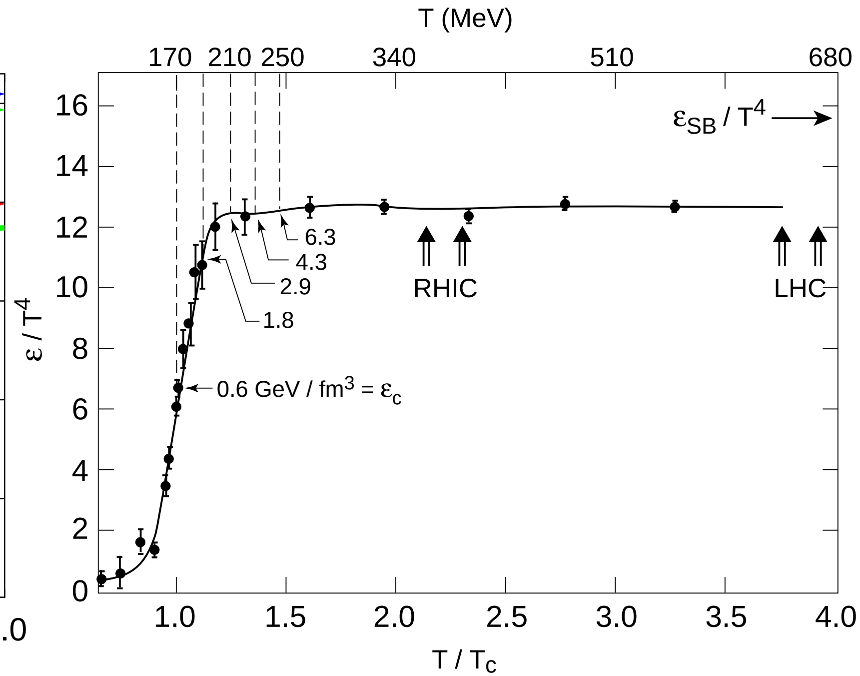
Best estimate for deconfining transition:

$$T_c \simeq 170(5)\text{MeV}$$

Equation of State at $\mu_B = 0$ ($L_t = 4$)

Bielefeld group (2000)





- For $N_f = 2$ transition is crossover
- For $N_f = 3$ and $m < m_c$ transition is first order
- For realistic "$N_f = 2 + 1$" a crossover currently favoured

NB
$$\frac{p_{SB}}{T^4} = \frac{8\pi^2}{45} + N_f \left[\frac{7\pi^2}{60} + \frac{1}{2}\left(\frac{\mu_q}{T}\right)^2 + \frac{1}{4\pi^2}\left(\frac{\mu_q}{T}\right)^4 \right]$$

The Sign Problem for $\mu \neq 0$

In Euclidean metric the QCD Lagrangian reads

$$\mathcal{L}_{QCD} = \bar{\psi}(M + m)\psi + \frac{1}{4}F_{\mu\nu}F_{\mu\nu}$$

with $M(\mu) = \not{D}[A] + \mu\gamma_0$

Straightforward to show $\gamma_5 M(\mu)\gamma_5 \equiv M^{\dagger}(-\mu) \Rightarrow$
$\det M(\mu) = (\det M(-\mu))^{*}$

ie. Path integral measure is not positive definite for $\mu \neq 0$

Fundamental reason is explicit breaking of time reversal symmetry

Monte Carlo importance sampling, the mainstay of lattice QCD, is ineffective

A formal solution to the Sign Problem is *reweighting* ie. to include the phase of the determinant in the observable:

$$\langle \mathcal{O} \rangle \equiv \frac{\langle\langle \mathcal{O} \,\arg(\det M)\rangle\rangle}{\langle\langle \arg(\det M)\rangle\rangle}$$

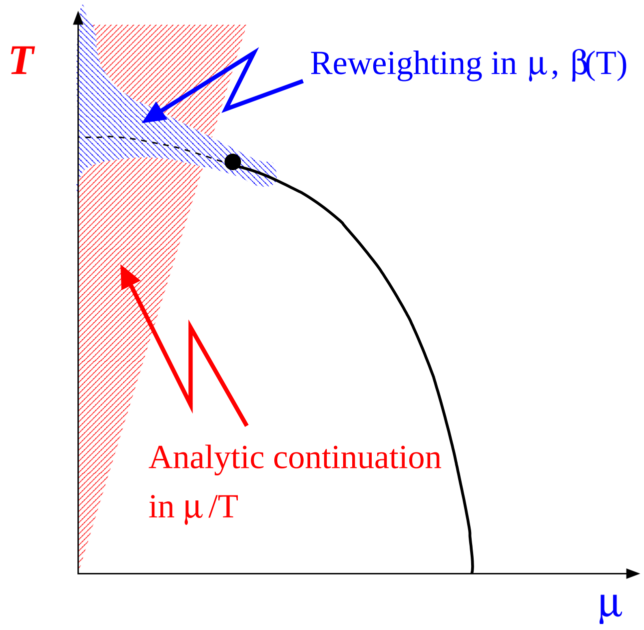
with $\langle\langle \ldots \rangle\rangle$ defined with a positive measure $|\det M| e^{-S_{boson}}$

Unfortunately both denominator and numerator are exponentially suppressed:

$$\langle\langle \arg(\det M)\rangle\rangle = \frac{\langle 1 \rangle}{\langle\langle 1 \rangle\rangle} = \frac{Z_{true}}{Z_{fake}} = \exp(-\Delta F) \sim \exp(-\#V)$$

Expect signal to be overwhelmed by noise in thermodynamic limit $V \to \infty$

Two Routes into the Plane



(I) Analytic continuation in μ/T by either

Taylor expansion @ $\mu = 0$

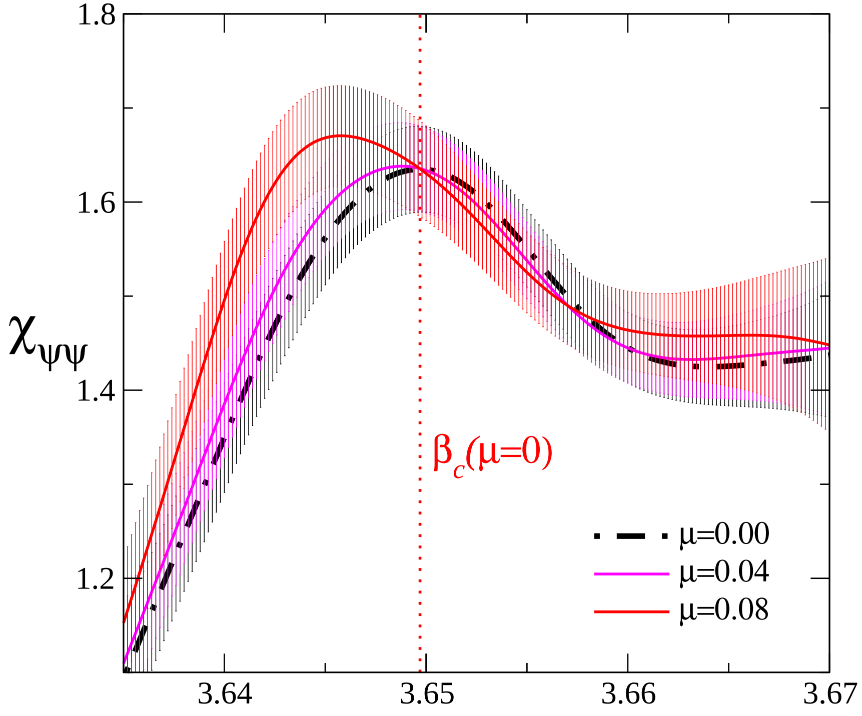
Gavai & Gupta; QCDTARO

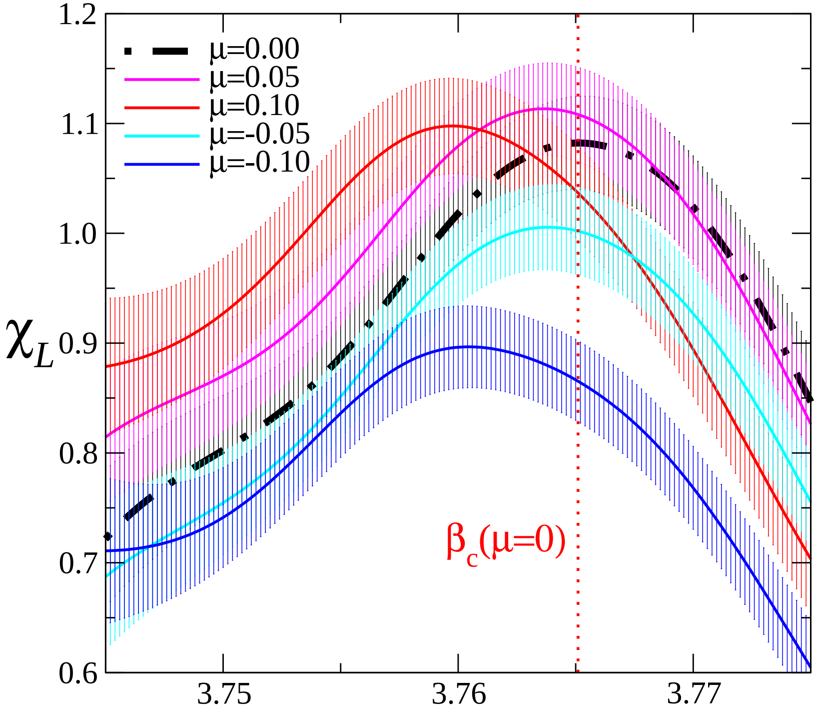
Simulation with imaginary $\tilde{\mu} = i\mu$ de Forcrand & Philipsen; d'Elia & Lombardo

effective for $\frac{\mu}{T} < \min\left(\frac{\mu_E}{T_E}, \frac{\pi}{3}\right)$

(II) Reweighting along transition line $T_c(\mu)$ Fodor & Katz

Overlap between (μ, T) and $(\mu + \Delta\mu, T + \Delta T)$ remains large, so multi-parameter reweighting unusually effective





The Bielefeld/Swansea group used a hybrid approach; ie. reweight using a Taylor expansion of the weight:

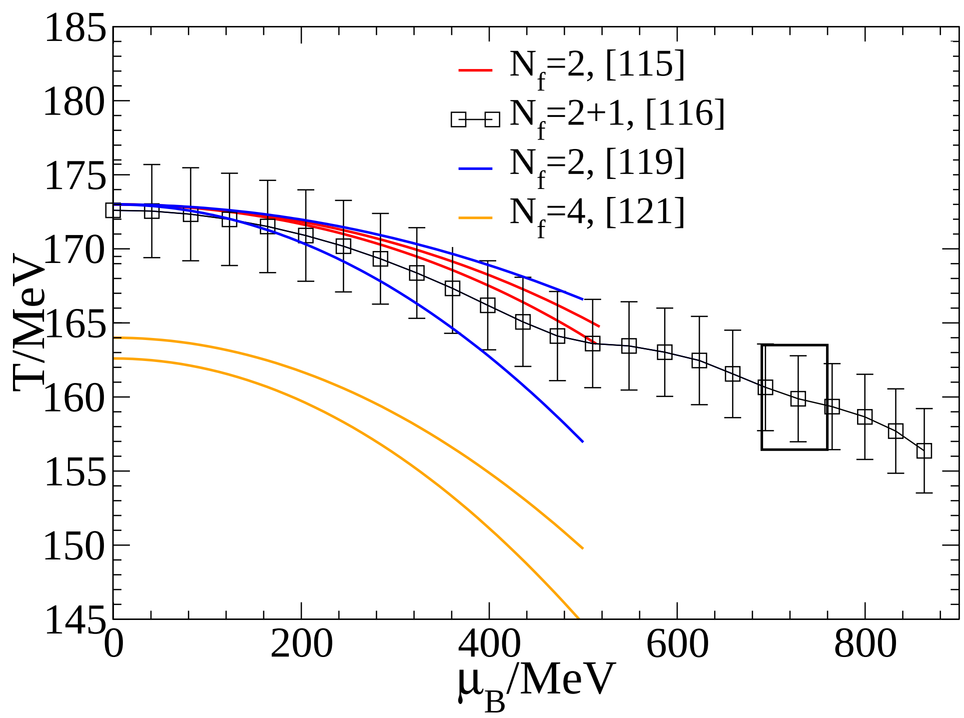
Allton *et al*, PRD66(2002)074507

$$\ln\left(\frac{\det M(\mu)}{\det M(0)}\right) = \sum_n \frac{\mu^n}{n!} \frac{\partial^n \ln \det M}{\partial \mu^n}\bigg|_{\mu=0}$$

This is relatively cheap and enables the use of large spatial volumes ($16^3 \times 4$ using $N_f = 2$ flavors of p4-improved staggered fermion).

Note with $L_t = 4$ the lattice is coarse: $a^{-1}(T_c) \simeq 700$MeV

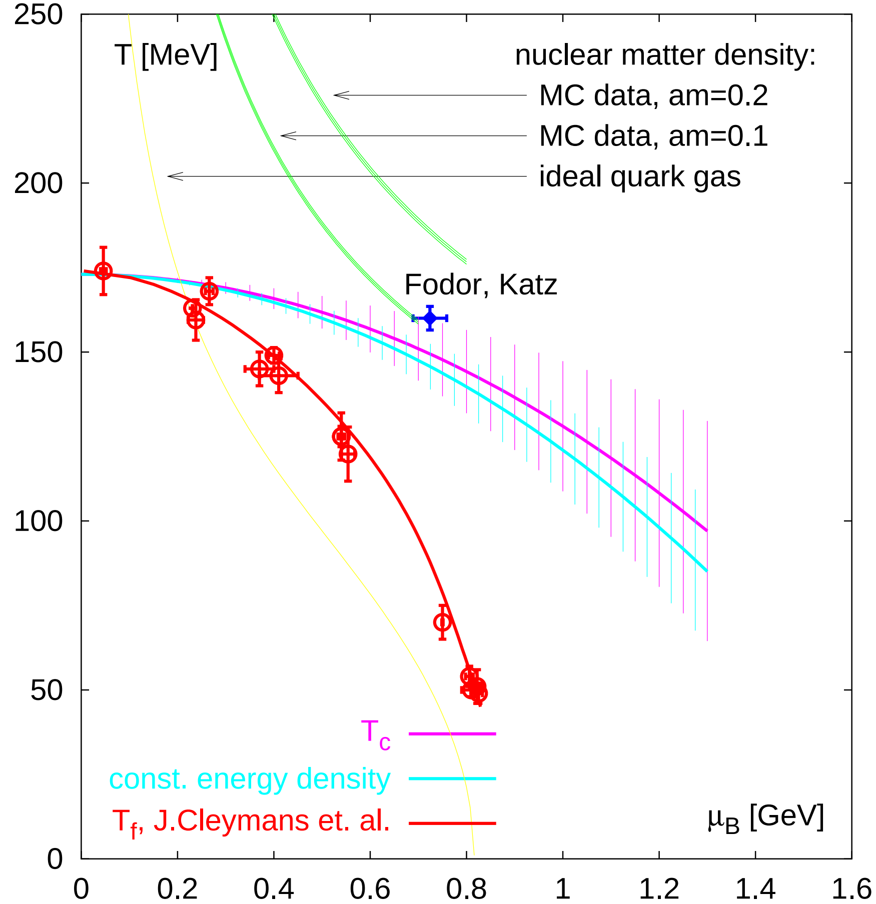
The (Pseudo)-Critical Line



[E. Laermann & O. Philipsen, Ann.Rev.Nucl.Part.Sci.53:163,2003]

Remarkable consensus on the curvature. . .

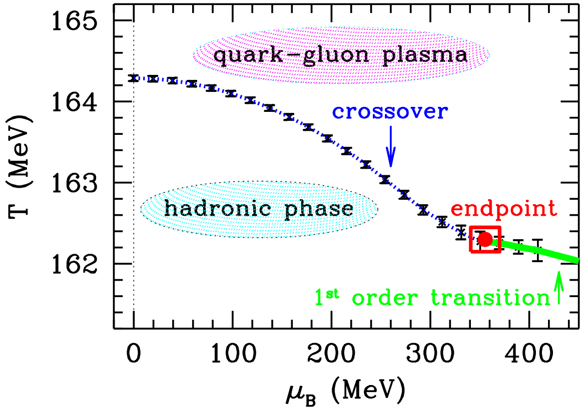
RHIC collisions operate in region $\mu_B \sim 45$MeV



The pseudocritical line found lies well above the (μ_B, T) trajectory marking chemical freezeout in RHIC collisions

$\Rightarrow$ is there a region of the phase diagram where *hadrons* interact very strongly (ie. inelastically)? <u>So what?</u>

The Critical Endpoint μ_E/T_E

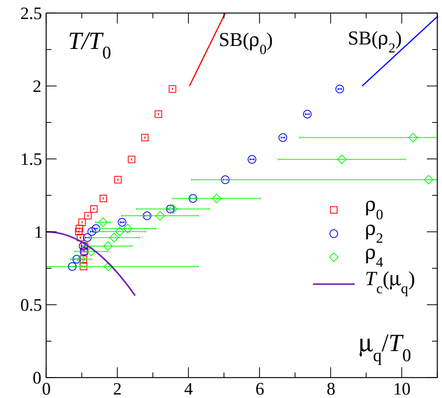


Reweighting estimate
via Lee-Yang zeroes
$$\mu_E/T_E = 2.2(2)$$
Z. Fodor & S.D. Katz JHEP0404(2004)050

Taylor expansion estimate from apparent radius of convergence

$$\mu_E/T_E \gtrsim |c_4/c_6| \quad \sim \quad 3.3(6)$$

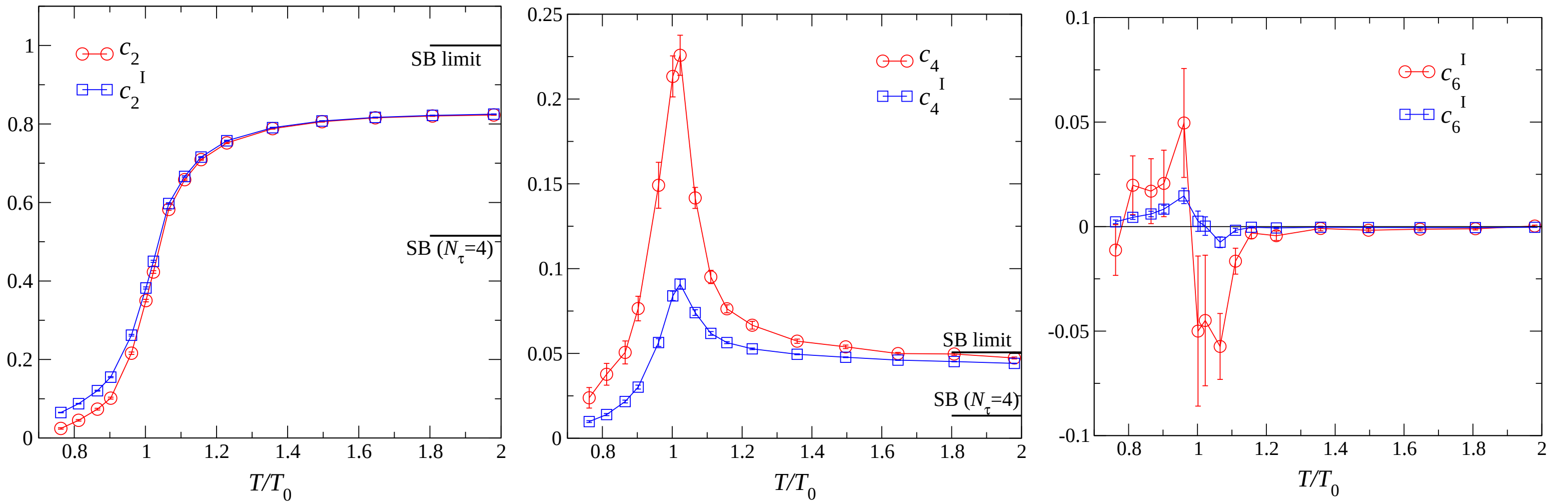
Allton *et al* PRD68(2003)014507



Analytic estimate via Binder cumulant $\langle(\delta\mathcal{O})^4\rangle/\langle(\delta\mathcal{O})^2\rangle^2$ evaluated at imaginary $\mu \Rightarrow \mu_E/T_E \sim O(20)$!

P. de Forcrand & O. Philipsen NPB673(2003)170

Taylor Expansion



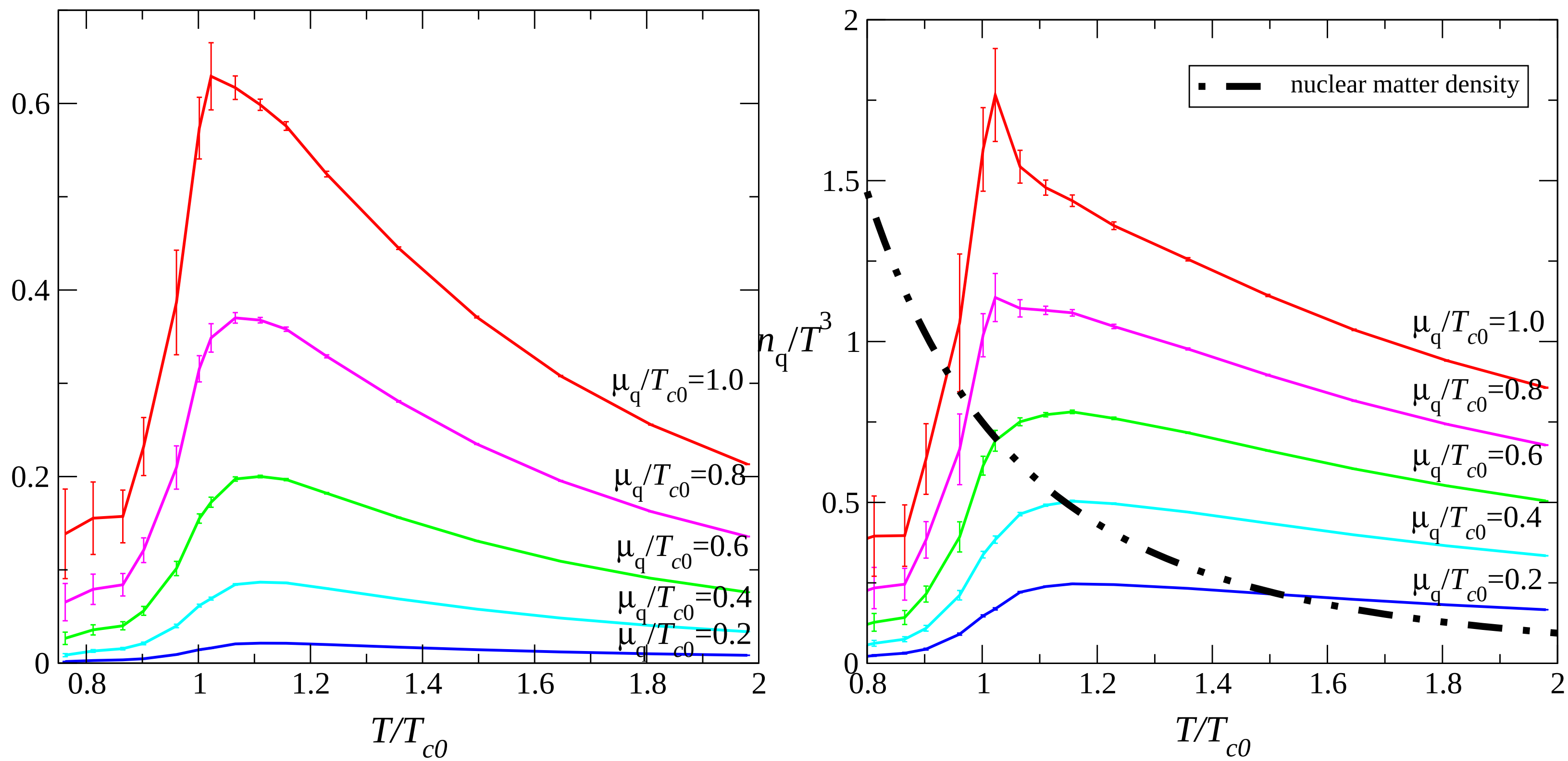
In our most recent work we develop the Taylor expansion of the free energy to $O((\mu_q/T)^6)$ **(recall $c_6^{SB} = 0$)**:

$$\frac{p}{T^4} = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu_q}{T}\right)^n \quad \text{with} \quad c_n(T) = \frac{1}{n!} \frac{\partial^n (p/T^4)}{\partial(\mu_q/T)^n}\bigg|_{\mu_q=0}$$

Similarly we define expansion coefficients

$$c_n^I(T) = \frac{1}{n!} \frac{\partial^n (p/T^4)}{\partial(\mu_I/T)^2 \partial(\mu_q/T)^{n-2}}\bigg|_{\mu_q=0,\mu_I=0}$$

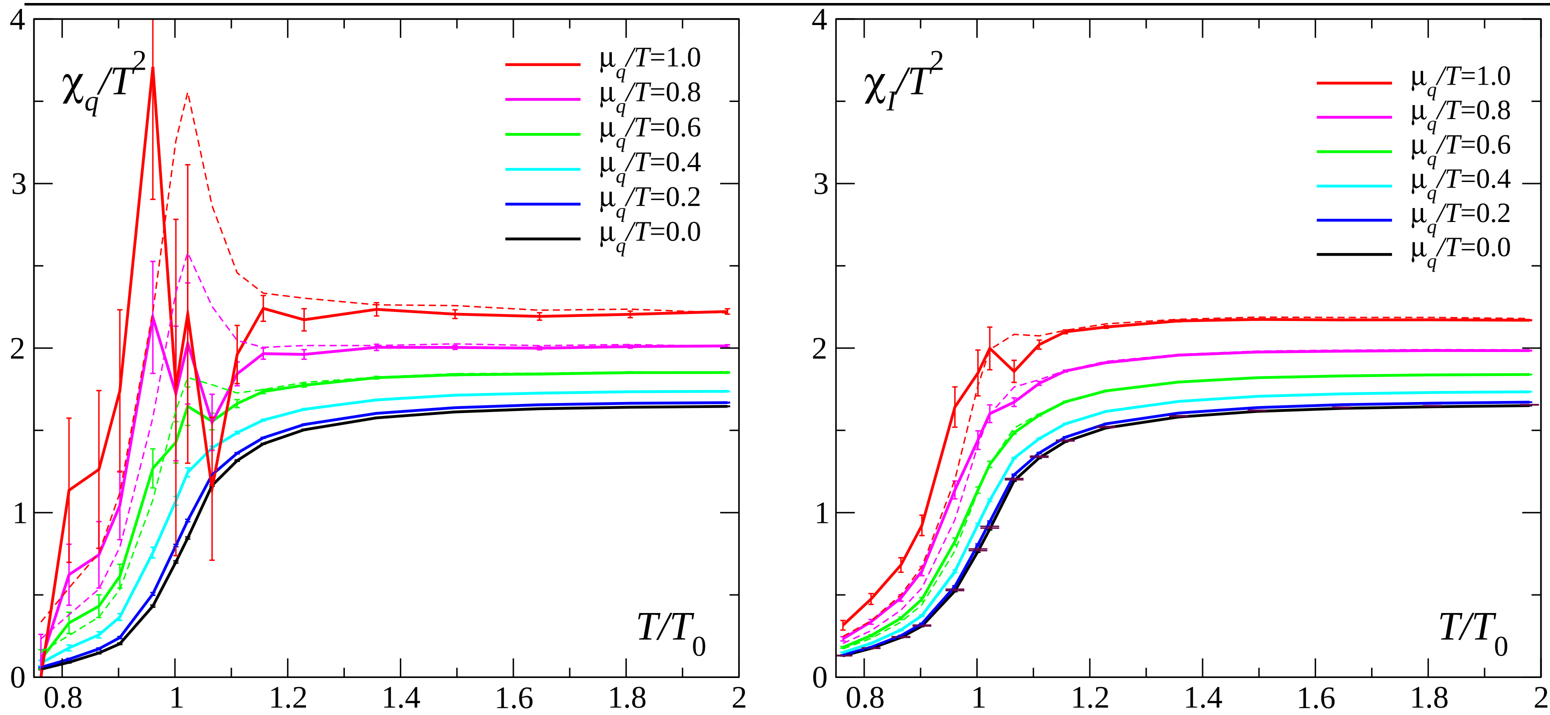
Equation of State Allton *et al* PRD68(2003)014507



Pressure change $\Delta p/T^4$ Quark density n_q/T^3

$$\Delta \frac{p(\mu,T)}{T^4} = \frac{p(\mu,T) - p(0,T)}{T^4} = \sum_{n=1}^{n_{max}} c_n(T) \left(\frac{\mu}{T}\right)^n \quad ; \qquad n_q = \frac{\partial p}{\partial \mu}$$

Growth of Baryonic Fluctuations

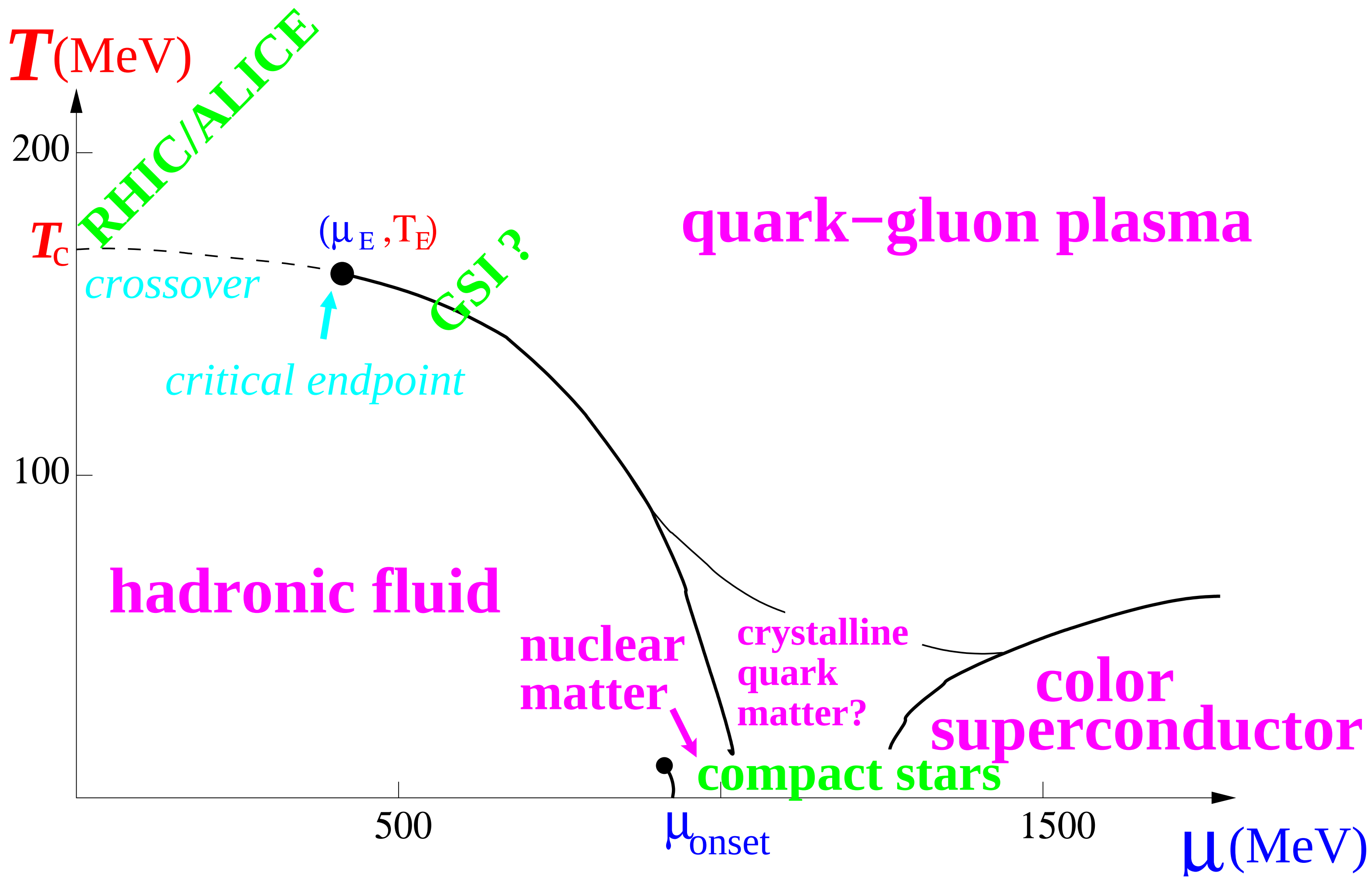


Allton *et al* PRD68(2003)014507

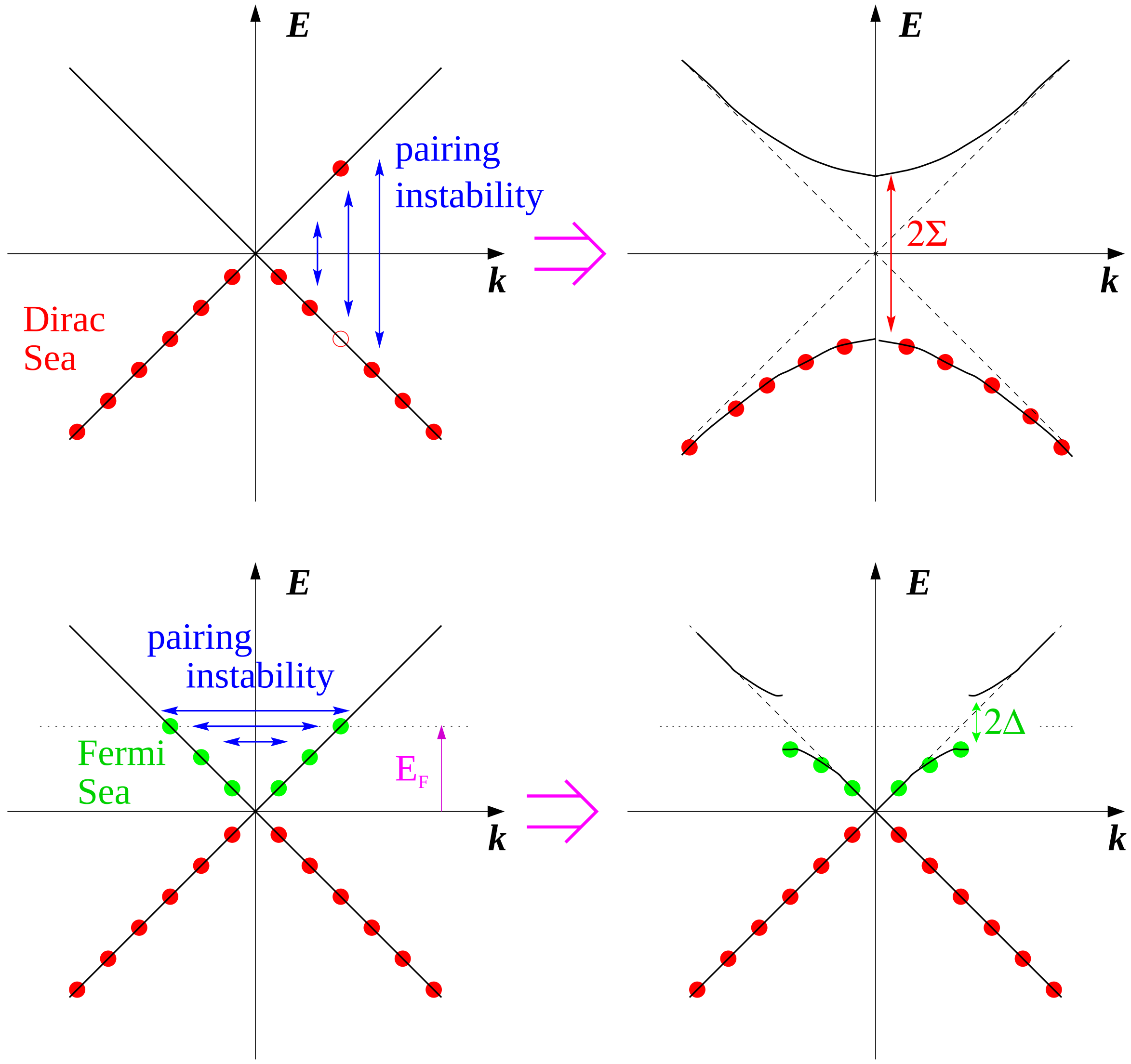
Quark number susceptibility $\chi_q = \frac{\partial^2 \ln Z}{\partial \mu_q^2}$ appears singular near $\mu_q/T \sim 1$; isospin susceptibility $\chi_I = \frac{\partial^2 \ln Z}{\partial \mu_I^2}$ does not

Massless field at critical point a combination of the Galilean scalar isoscalars $\bar{\psi}\psi$ and $\bar{\psi}\gamma_0\psi$?

The QCD Phase Diagram



χSB vs. Cooper Pairing



Color Superconductivity

In the asymptotic limit $\mu \to \infty$, $g(\mu) \to 0$, the ground state of QCD is the *color-flavor locked* (CFL) state characterised by a BCS instability, [D. Bailin and A. Love, Phys.Rep. 107(1984)325]
ie. diquark pairs at the Fermi surface condense via

$$\langle q_i^\alpha(p) C\gamma_5 q_j^\beta(-p)\rangle \sim \varepsilon^{A\alpha\beta}\varepsilon_{Aij} \times \text{const.}$$

breaking $\mathrm{SU}(3)_c \otimes \mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R \otimes \mathrm{U}(1)_B \otimes \mathrm{U}(1)_Q$
$\longrightarrow \mathrm{SU}(3)_\Delta \otimes \mathrm{U}(1)_{\tilde{Q}}$

The ground state is simultaneously
superconducting (8 gapped gluons, ie. get mass $O(\Delta)$),

superfluid (1 Goldstone),

and *transparent* (all quasiparticles with $\tilde{Q} \neq 0$ gapped).
[M.G. Alford, K. Rajagopal and F. Wilczek, Nucl.Phys.B537(1999)443]

The NJL Model

Effective description of soft pions interacting with nucleons

$$\mathcal{L}_{NJL} = \bar{\psi}(\partial\!\!\!/ + m + \mu\gamma_0)\psi - \frac{g^2}{2}[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\vec{\tau}\psi)^2]$$
$$\sim \bar{\psi}(\partial\!\!\!/ + m + \mu\gamma_0 + \sigma + i\gamma_5\vec{\pi}.\vec{\tau})\psi + \frac{2}{g^2}(\sigma^2 + \vec{\pi}.\vec{\pi})$$

Introduce isopsin indices so full global symmetry is SU(2)$_L\otimes$SU(2)$_R\otimes$U(1)$_B$

Dynamical χSB for $g^2 > g_c^2 \Rightarrow$ isotriplet Goldstone $\vec{\pi}$

Scalar isoscalar diquark $\psi^{tr}C\gamma_5 \otimes \tau_2 \otimes A^{color}\psi$ breaks U(1)$_B$

$\Rightarrow$ diquark condensation signals high density ground state is superfluid

Model is renormalisable in 2+1d so GN analysis holds

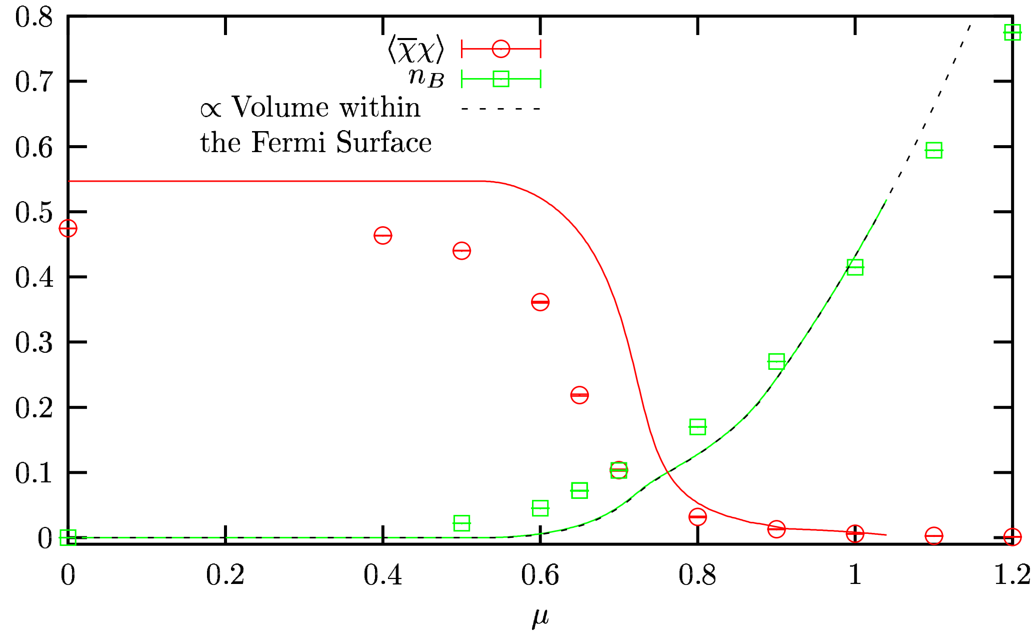
In 3+1d, an explicit cutoff is required. We follow the large-N_f (Hartree) approach of Klevansky (1992) and match lattice parameters to low energy phenomenology:

Phenomenological Observables fitted	Lattice Parameters extracted
$\Sigma_0 = 400$MeV	$ma = 0.006$
$f_\pi = 93$MeV	$1/g^2 = 0.495$
$m_\pi = 138$MeV	$a^{-1} = 720$MeV

The lattice regularisation preserves

$$\text{SU(2)}_L \otimes \text{SU(2)}_R \otimes \text{U(1)}_B$$

Equation of State and Diquark Condensation



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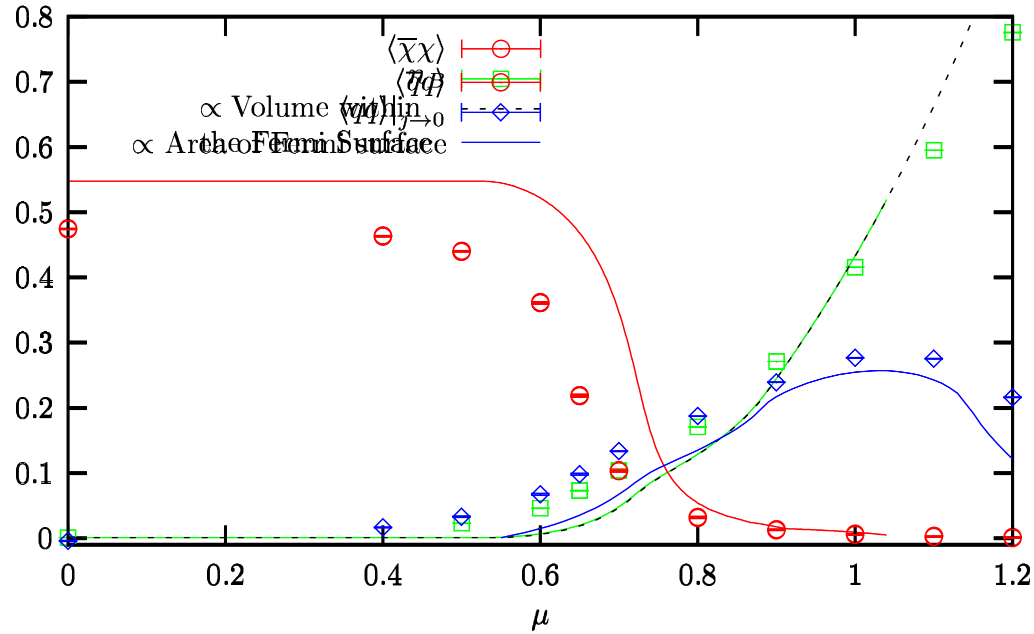
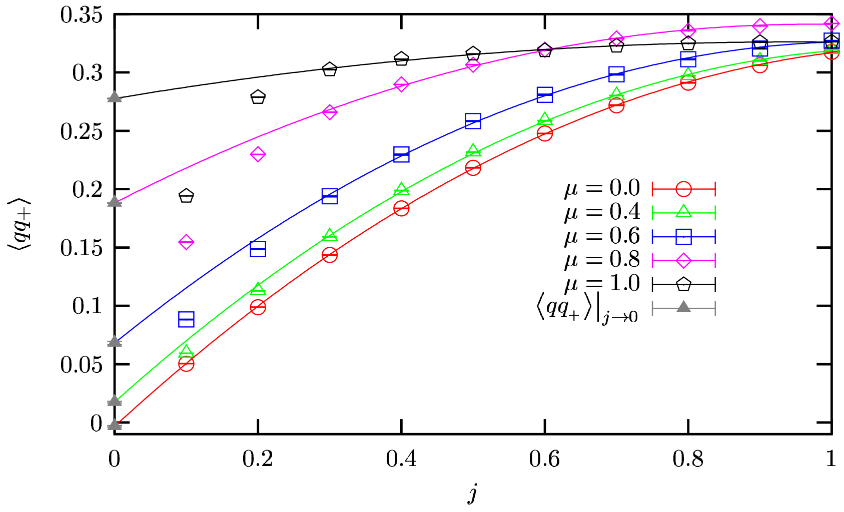




Add source $j[\psi^{tr}\psi + \bar{\psi}\bar{\psi}^{tr}]$

Diquark condensate estimated by taking $j \to 0$

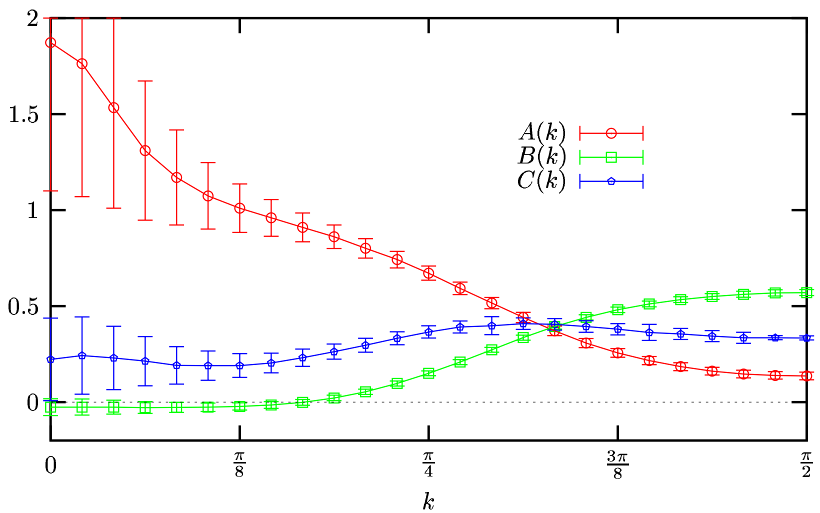
Our fits exclude $j \leq 0.2$

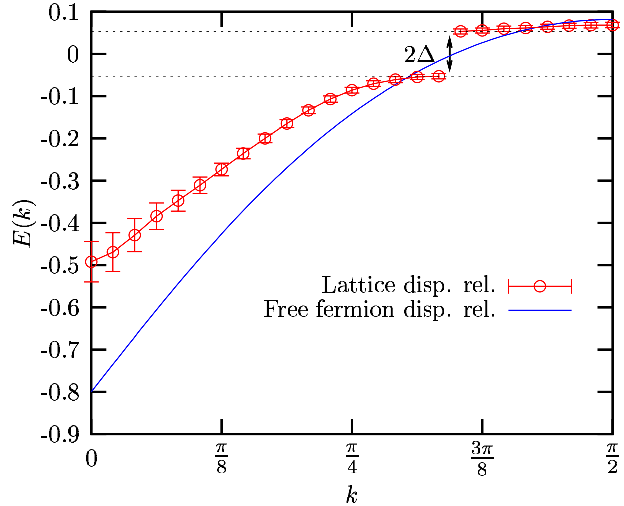
The Superfluid Gap

Quasiparticle propagator:

$$\langle \psi_u(0)\bar{\psi}_u(t)\rangle = Ae^{-Et} + Be^{-E(L_t-t)}$$
$$\langle \psi_u(0)\psi_d(t)\rangle = C(e^{-Et} - e^{-E(L_t-t)})$$

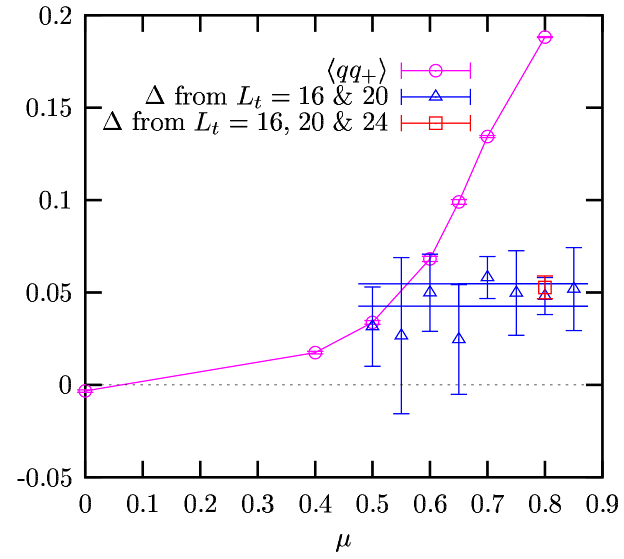
Results from $96 \times 12^2 \times L_t$, $\mu a = 0.8$ extrapolated to $L_t \to \infty$ (ie. $T \to 0$) then $j \to 0$





The gap at the Fermi surface signals superfluidity

SJH & D.N. Walters PLB548(2002)196 PRD69(2004)076011



- $\Delta/\Sigma_0 \simeq 0.15 \Rightarrow \Delta \simeq 60$MeV
in agreement with self-consistent approaches

- Similar formalism to study non-relativistic model for
EITHER nuclear matter (with or without pions)
$\Rightarrow$ calculation of E/A D. Lee & T. Schäfer `nucl-th/0412002`
OR Cold atoms with tunable scattering length
$\Rightarrow$ study of BEC/BCS crossover M. Wingate `cond-mat/0502372`

In either case non-perturbative due to large dimensionless parameter $k_F|a| \gg 1$, with a the s-wave scattering length.

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And finally. . .